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Universal Functions in Euclidean Quantum Gravity

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Abstract. A key problem in the attempt to quantize the gravitational field is the choice of boundary conditions. These are mixed, in that spatial and normal components of metric perturbations obey different sets of boundary conditions. In the covariant quantization scheme this leads to a boundary operator involving both normal and tangential derivatives of metric perturbations. On studying the corresponding heat-kernel asymptotics, one finds that universal, tensorial, nonpolynomial structures contribute through the integrals over the boundary of linear combinations of all geometric invariants of the problem. These universal functions are independent of conformal rescalings of the background metric, and they might lead to a deep revolution in the current understanding of quantum gravity.

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The division of physics into differential equations for the fields, and boundary conditions for the solutions of such equations, has proved to be very useful both in cosmology and in the current understanding of (quantized) field theories. In particular, in the sum-over-histories formulation of quantum gravity, an important task is to give a precise definition of the $\langle \text{out} | \text{in} \rangle$ amplitudes of going from suitable in-data on an initial spacelike three-surface, to suitable out-data on a final spacelike three-surface. One can then try to perform a semiclassical evaluation of the path-integral. Despite the well known lack of perturbative renormalizability of a quantum gravity theory based on the Einstein's action, one can actually discover a lot of new exciting properties even just from a careful analysis of the one-loop semiclassical approximation. Our essay is devoted to this problem for pure gravity in four dimensions, when a compact Riemannian four-manifold, (M, g) , with boundary, ∂M , is studied. Although no further assumption is made, the reader should be aware that our work corresponds to the Euclidean approach to quantum cosmology, rather than to the analysis of $\langle \text{out} | \text{in} \rangle$ amplitudes with the associated asymptotic regions.

To begin, let us assume that spatial components of metric perturbations, say h_{ij} , are set to zero at the boundary:

$$\left[h_{ij} \right]_{\partial M} = 0 \quad . \quad (1)$$

Of course, this is suggested by what one can do in linearized theory at the classical level. A basic ingredient in our analysis is that (1) should be preserved under infinitesimal diffeomorphisms on metric perturbations (here, ∇ is the Levi-Civita connection of the background):

$$\varphi h_{ab} = h_{ab} + \nabla_{(a} \varphi_{b)} \quad . \quad (2)$$

Their action on h_{ij} reads

$$\varphi h_{ij} = h_{ij} + \widehat{\nabla}_{(i} \varphi_{j)} + K_{ij} \varphi_0 \quad , \quad (3)$$

where φ_b is the ghost one-form, K_{ij} is the extrinsic-curvature tensor of ∂M , and $\widehat{\nabla}$ denotes three-dimensional covariant differentiation tangentially with respect to the Levi-Civita connection of the boundary. It is then clear that, if K_{ij} does not vanish, a necessary and sufficient condition for the preservation of the boundary conditions (1) under the

transformations (3) is that the whole ghost one-form vanishes on the boundary, i.e.

$$\left[\varphi_a \right]_{\partial M} = 0 \quad . \quad (4)$$

At this stage, the remaining set of boundary conditions on metric perturbations, whose invariance under infinitesimal diffeomorphisms (2) is guaranteed by (4), involves setting to zero at the boundary a linear gauge-averaging functional in the Faddeev-Popov scheme:

$$\left[\Phi_a(h) \right]_{\partial M} = 0 \quad . \quad (5)$$

What happens is that, under the transformations (2), one finds

$$\Phi_a({}^\varphi h) = \Phi_a(h) + \mathcal{F}_a{}^b \varphi_b \quad , \quad (6)$$

where $\mathcal{F}_a{}^b$ is the ghost operator. By expanding then the ghost one-form into a complete orthonormal set of eigenfunctions of $\mathcal{F}_a{}^b$ with Dirichlet boundary conditions (4), one sees that the boundary conditions (5) are invariant under (2).

In particular, if a covariant gauge-averaging functional of the de Donder type is used, i.e.

$$\Phi_a(h) \equiv \nabla^b \left(h_{ab} - \frac{1}{2} g_{ab} g^{cd} h_{cd} \right) \quad , \quad (7)$$

the boundary conditions (5) include both normal and tangential derivatives of the normal components h_{00} and h_{0i} . It is then possible to express both (1) and (5) by a single equation

$$\left[\mathcal{B}h \right]_{\partial M} = 0 \quad , \quad (8)$$

where \mathcal{B} is the boundary operator, defined as (\mathbb{I} denotes $\delta^c{}_{(a} \delta^d{}_{b)}$)

$$\mathcal{B} \equiv (\mathbb{I} - \Pi) \left(H \nabla_n + \Gamma^i \widehat{\nabla}_i + S \right) + \mu \Pi \quad . \quad (9)$$

With our notation, n^b denotes the normal to ∂M , $\nabla_n \equiv n^a \nabla_a$ is the normal derivative, μ is a dimensional parameter, and q, Π, Γ^i and S are tensors defined by

$$q_{ab} \equiv g_{ab} - n_a n_b \quad , \quad (10)$$

$$\Pi_{ab}{}^{cd} \equiv q^c{}_{(a} q^d{}_{b)} \quad , \quad (11)$$

$$H^{ab\ cd} \equiv g^{a(c} g^{d)b} - \frac{1}{2} g^{ab} g^{cd} \quad , \quad (12)$$

$$\Gamma^i_{ab\ cd} \equiv n_a n_b e^{i(c} n^{d)} - n_{(a} e^i_{b)} n^c n^d \quad , \quad (13)$$

$$S_{ab\ cd} \equiv -n_a n_b n^c n^d + 2n_{(a} e^i_{b)} e^{j(c} n^{d)} \left[K_{ij} - \gamma_{ij} (\text{tr} K) \right] \quad , \quad (14)$$

where e^i_a is a local basis of one-forms on the boundary, and $\gamma^{ij} = g^{ab} e^i_a e^j_b$ is the induced metric on the boundary.

The form (9) of the boundary operator is not generic, but depends, as we said, on the choice (7) for the gauge-averaging functional, jointly with (1) and (5). We are thus studying just one of the possible schemes for mixed boundary conditions in Euclidean quantum gravity, following the work in Refs. [1–6]. Our form of the boundary operator has been also obtained in Ref. [7], within the framework of Becchi-Rouet-Stora-Tyutin invariant boundary conditions in quantum field theory. Note that the matrix $\Gamma^2 \equiv \gamma_{ij} \Gamma^i \Gamma^j$, which, from (13), reads [6]

$$\Gamma^2 = -\frac{3}{2} n_a n_b n^c n^d - n_{(a} q^{(c}_{b)} n^{d)} \quad , \quad (15)$$

commutes with S :

$$\Gamma^2 S = S \Gamma^2 = \frac{3}{2} n_a n_b n^c n^d - n_{(a} e^i_{b)} e^{j(c} n^{d)} \left[K_{ij} - \gamma_{ij} (\text{tr} K) \right] \quad . \quad (16)$$

However, Γ^2 does not commute with the matrices Γ^i . Indeed, the explicit calculation shows that

$$\Gamma^2 \Gamma^i = -\frac{3}{2} n_a n_b e^{i(c} n^{d)} + \frac{1}{2} n_{(a} e^i_{b)} n^c n^d \quad , \quad (17)$$

whereas

$$\Gamma^i \Gamma^2 = -\frac{1}{2} n_a n_b e^{i(c} n^{d)} + \frac{3}{2} n_{(a} e^i_{b)} n^c n^d \quad . \quad (18)$$

These remarks are of crucial importance for the following reasons. A similar form of the boundary operator (9), when Π does not occur and $H = \mathbb{I}$, i.e.

$$\mathcal{B} = \nabla_n + \Gamma^i \widehat{\nabla}_i + \frac{1}{2} (\widehat{\nabla}_i \Gamma^i) + S \quad , \quad (19)$$

for a general second-order operator P of Laplace type (∇ being a connection and E some endomorphism of a vector bundle V):

$$P \equiv -g^{ab} \nabla_a \nabla_b - E \quad , \quad (20)$$

was studied in Ref. [2]. In general, the matrices Γ^i and S do not have the form (13) and (14) but satisfy the conditions $\Gamma^{i\dagger} = -\Gamma^i$, $S^\dagger = S$ [2,6].

In the corresponding asymptotic expansion of the integrated heat-kernel [6], it is convenient to introduce a smooth function on M , say f , which makes it possible to recover the distributional behaviour of the heat-kernel near the boundary. One then finds in four dimensions, as $t \rightarrow 0^+$, the asymptotics

$$\mathrm{Tr}_{L^2}\left(fe^{-tP}\right) \sim (4\pi t)^{-2} \sum_{n=0}^{\infty} t^{n/2} a_{n/2}(f, P) \quad , \quad (21)$$

where the coefficients $a_{n/2}(f, P)$ are obtained by integrating geometric invariants over M (interior terms) and ∂M (boundary terms). More precisely, the consideration of all the invariants which can be built from the operators P (20) and \mathcal{B} (19) shows that the first two coefficients in (21) can be written in the form [2,6]

$$a_0(f, P) = \int_M \mathrm{Tr}(f) \quad , \quad (22)$$

$$a_{1/2}(f, P) = \int_{\partial M} \mathrm{Tr}(\rho(\Gamma)f) \quad . \quad (23)$$

The occurrence of Γ^i in the boundary operator (19) leads to many additional invariants, further to the standard contributions for Dirichlet or Neumann boundary conditions, in the general formulae for higher-order heat-kernel coefficients $a_{n/2}(f, P)$. In particular, when Γ^2 commutes with Γ^i and S , one can control all additional contributions. The coefficient a_1 has then the form [2,6]

$$\begin{aligned} a_1(f, P) = & \int_M \mathrm{Tr}\left[f(\alpha_1 E + \alpha_2 R^a{}_a)\right] \\ & + \int_{\partial M} \mathrm{Tr}\left[f\left(b_0(\Gamma)(\mathrm{tr}K) + \sigma_1(\Gamma)K_{ij}\Gamma^i\Gamma^j + b_2(\Gamma)S\right) + b_1(\Gamma)\nabla_n f\right] \quad . \end{aligned} \quad (24)$$

Remarkably, while the parameters α_1 and α_2 are universal constants, $\rho(\Gamma)$, $\sigma_1(\Gamma)$, $b_0(\Gamma)$, $b_1(\Gamma)$ and $b_2(\Gamma)$ turn out to be *universal functions*. This means that they are functions of position on the boundary, and their dependence on Γ^i is realized through analytic functions of Γ^2 . Thus, by construction, these functions are independent of conformal rescalings of the background metric. Moreover, all parameters in (24) are also independent of the dimension

of M . Further (assuming for simplicity $\widehat{\nabla}_i \Gamma^j = 0$), to obtain the form of $a_{3/2}(f, P)$, one has to consider all possible contractions of the matrices Γ^i (together with the normal n^a and the metric γ^{ij}) with geometric objects that can be put symbolically in the form

$$fKK \ , \ fKS \ , \ f\widehat{\nabla}K \ , \ f\widehat{\nabla}S \ , \ fR \ , \ fF$$

and $K\nabla_n f$ (here, R and F denote the Riemann curvature of M , and the curvature of the bundle connection, respectively). Similarly, the general form of $a_2(f, P)$ receives contributions from all contractions of Γ^i , n^a and γ^{ij} with geometric terms of the form:

$$fKKK \ , \ fKKS \ , \ fKSS \ , \ fRK \ , \ fFK \ , \ fEK \ , \ fRS \ , \ fFS \ ,$$

$$fK\widehat{\nabla}K \ , \ fS\widehat{\nabla}K \ , \ fK\widehat{\nabla}S \ , \ fS\widehat{\nabla}S \ , \ f\widehat{\nabla}\widehat{\nabla}K \ ,$$

$$f\widehat{\nabla}\widehat{\nabla}S \ , \ f\nabla R \ , \ f\nabla F \ , \ f\nabla E \ ,$$

as well as

$$KK\nabla_n f \ , \ KS\nabla_n f \ , \ (\widehat{\nabla}K)\nabla_n f \ , \ (\widehat{\nabla}S)\nabla_n f \ , \ R\nabla_n f \ , \ F\nabla_n f$$

and $K\nabla_n \nabla_n f$.

In our essay, however, we are concerned with Euclidean quantum gravity in four dimensions (four dimensions being more relevant for the world we live in). Thus, focusing on our original problem, we have to point out that a more basic difficulty occurs.

In the simple case considered in Refs. [2] and [6], when the matrices Γ^2 and Γ^i commute, all tensor structures of rank m built from Γ^i have the form

$$T^{i_1 \dots i_m}(\Gamma^j) = T(\Gamma^2) \Gamma^{i_1} \dots \Gamma^{i_m} \ , \quad (25)$$

where $T(\Gamma^2)$ are universal functions of Γ^2 only. As we have seen, this is not the case, however, for the gravitational field, since the matrices Γ^2 and Γ^i do not commute (see (17) and (18)). This means that the tensor structures $T^{i_1 \dots i_m}(\Gamma^j)$ on the boundary, formed by the matrices Γ^i , do not have the simple form (25). In particular, even the scalar functions, $\alpha(\Gamma^j)$, cannot be always presented as the trace of functions of Γ^2 only, and the second-rank

tensors $T^{ij}(\Gamma^l)$ are not polynomial in Γ^i . For example, there are infinitely many different tensors of the type

$$T_{(m)}^{ij}(\Gamma^l) \equiv \text{Tr} \{ \alpha_{(m)}(\Gamma^2) \Gamma^i \beta_{(m)}(\Gamma^2) \Gamma^j \} \quad ,$$

which can contribute already to a_1 . Thus, for the generalized boundary operator (9) with arbitrary noncommuting matrices Γ^i , even the coefficient a_1 is unknown.

One thus faces a highly nontrivial problem. On one hand, analytic results exist for the a_2 coefficient with boundary operator (9) in the particular case of a flat Euclidean background bounded by a three-sphere [3,5]. Moreover, it has been shown in Ref. [4] that the boundary operator (9) leads to a self-adjoint boundary-value problem on metric perturbations. However, in the noncommuting case relevant for gravity, even the building blocks of geometric invariants involving Γ^i are unknown. That is why it remains unclear how to write a general and unambiguous formula for heat-kernel coefficients. The solution of this problem is of the greatest importance in quantum gravity for the following reasons:

- (i) to improve the understanding of BRST invariant boundary conditions [7];
- (ii) to obtain an entirely geometric description of one-loop divergences in quantum gravity and quantum supergravity [5];
- (iii) as a first step towards the quantization in arbitrary gauges on manifolds with boundary;
- (iv) to clarify the differences between Yang-Mills fields and the gravitational field;
- (v) to complete the application of the effective-action programme to perturbative quantum gravity.

Further to this, we would like to end by emphasizing that the problem remains of how to build and use a non-perturbative theory of the effective action in quantum gravity. This investigation, jointly with the analysis of universal functions of the quantized gravitational field, would lead to an entirely new vision, showing how deep is the impact of boundary conditions and effective-action methods on the attempt to combine the ideas of quantum physics and general relativity.

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